International Journal of Research Studies in Science, Engineering and Technology
Volume 8, Issue 1, 2021, PP 30-37
ISSN : 2349-476X
DOI : https://doi.org/10.22259/2349-476X.0801004

On Λ-Fractional Field Theorems

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ABSTRACT

Fractional field theorems are developed in the Λ-fractional space, introduced by the recently proposed Λ-fractional derivative. Since Λ-fractional derivative exhibits all the properties of a common derivative in the Λ-fractional space, the conventional field theorems, that are valid in the Λ-fractional space, are pulled back in the initial space. Furthermore, Λ-fractional calculus of variations will be discussed with application to a vibrating string.

Keywords: *Λ-fractional space, Λ-fractional derivative, fractional field theorems, fractional variational methods, vibrating string.*

INTRODUCTION

Fractional calculus is a robust and intensely active mathematical field. Fractional derivatives where originated by Leibniz [1], Riemann [2], Liouville [3] and developed by many other scientists (Machado et al. [4] have written a very interesting article on fractional calculus history). The main advantage of this topic is that it describes globally various phenomena with temporal or spatial dependence.

On the other hand it seems to be a strong connection between fractional calculus and fractal geometry (Tatom [5]). Therefore that discipline is applied in many scientific areas, especially in physics [6,7,8,9,10] and mechanics [11,12,13,14].

As far as mechanics is concerned, viscoelasticity is a favourite theme for fractional calculus. That subject shows temporal dependence and therefore is very suitable for fractional mathematical models.

Many researches worked on that topic with success [15],[16]. On the other hand, Lazopoulos was the first who introduced Fractional derivatives in spatial dependent descriptions of materials [11].

The most famous fractional derivatives are Grünwald-Letnikov, Riemann-Lowville and Caputo derivative [8].

These mathematical operators thought face a serious problem: They do not fulfill the properties of a derivative according to differential topology. Therefore many researchers tried either to tackle that problem, or prove that those derivatives cannot fulfill the necessary requirements in any way [17,18]. Lazopoulos and al. [12] proposed Leibnitz' Fractional derivative that had already been introduced by Tarasov [23], in an effort to formulate fractional differential.

Nevertheless, that derivative could not comply with the rest of requirements of differential topology, so they introduced a more efficient edition of L-fractional derivative, Λ-fractional derivative. The latter showed all the properties of a proper derivative and therefore it is suitable for mathematical analysis [19].

In the present article, field theory is studied with the help of Λ-fractional derivative. At first a brief introduction of the derivative is presented, then the Λ-fractional space is presented, where that derivative behaves as a conventional derivative, and finally the most fundamental field theorems are described with the help of that derivative.

Then the results are pulled back to the initial space. Furthermore variational procedures with fractional multiple integrals are discussed with application to a vibrating string.

THE Λ-FRACTIONAL DERIVATIVE

A very brief outline of fractional calculus will be presented in the present chapter, while the interested reader is referred to refs.[6-10] for further information.

In fact the left and right fractional integrals, for a real $0<\gamma\leq 1$ are defined by,

$$ {}_{a}I_{x}^{\gamma} f(x) = \frac{1}{\Gamma(\gamma)} \int_{a}^{x} \frac{f(s)}{(x-s)^{1-\gamma}} ds \tag{1} $$

$$ {}_{x}I_{b}^{\gamma} f(x) = \frac{1}{\Gamma(\gamma)} \int_{x}^{b} \frac{f(s)}{(s-x)^{1-\gamma}} ds \tag{2} $$

γ is the order of fractional integrals with where Γ(x)=(x-1)! with Γ(γ) Euler's Gamma function.

Further, the left Riemann-Liouville fractional derivative is defined by:

$$ {}_{\alpha}^{RL}D_{x}^{\gamma} f(x) = \frac{d}{dx}\left({}_{\alpha}I_{x}^{1-\gamma}\left(f(x)\right)\right) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_{\alpha}^{x} \frac{f(s)}{(x-s)^{\gamma}} ds \tag{3} $$

whereas the right Riemann-Liouville derivative is defined by:

$$ {}_{x}^{RL}D_{b}^{\gamma} f(x) = \frac{d}{dx}\left({}_{x}I_{b}^{1-\gamma}\left(f(x)\right)\right) = -\frac{1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_{x}^{b} \frac{f(s)}{(s-x)^{\gamma}} ds \tag{4} $$

Let us point out that for the left fractional integrals and derivatives

$$ {}_{a}^{RL}D_{x}^{\gamma}\left({}_{a}I_{x}^{\gamma} f(x)\right) = f(x) \quad . \tag{5} $$

Similar relation is valid for the right Caputo derivative and right fractional integral.

The L-fractional derivative (L-FD) has been defined as

$$ {}_{a}^{\Lambda}D_{x}^{\gamma} f(x) = \frac{{}_{\alpha}^{RL}D_{x}^{\gamma} f(x)}{{}_{a}^{RL}D_{x}^{\gamma} x} \tag{6} $$

Recalling the definition of the Riemann-Liouville fractional derivative, Eq.(3), the Λ-FD is expressed by,

$$ {}_{a}^{\Lambda}D_{x}^{\gamma} f\left(x\right) = \frac{\dfrac{d\,{}_{a}I_{x}^{1-\gamma} f(x)}{dx}}{\dfrac{d\,{}_{a}I_{x}^{1-\gamma} x}{dx}} = \frac{d\,{}_{a}I_{x}^{1-\gamma} f(x)}{d\,{}_{a}I_{x}^{1-\gamma} x} \quad . \tag{7} $$

Considering as

$$ X = {}_{a}I_{x}^{1-\gamma} x \quad \text{and} \quad F(X) = {}_{a}I_{x}^{1-\gamma} f(x), \tag{8} $$

The Λ-FD appears to behave as a conventional derivative in the fractional Λ-space (X, F(X)) with local properties. In fact the Fractional Differential Geometry may be developed as a conventional differential geometry in the Λ-fractional space, (X, F(X)).

Indeed, Eq.(8a) yields

$$ X = \frac{x^{2-\gamma}}{\Gamma(3-\gamma)} \tag{9} $$

Further, Eqs.(8b,9) suggest that:

$$ F(x) = {}_{\alpha}I_{x}^{1-\gamma} f(x) = \frac{1}{\Gamma(1-\gamma)} \int_{\alpha}^{x} \frac{f(s)}{(x-s)^{\gamma}} ds \tag{10} $$

Inverting Eq.(9) it appears,

$$ x = \left(\Gamma(3-\gamma)X\right)^{1/(2-\gamma)} = x(X) \tag{11} $$

Proceeding further to the definition of the fractional Λ-space, inserting x(X) into Eq.(10), the function F(x) may be expressed as a function of X.

$$ F(X) = F\left(x(X)\right) \tag{12} $$

Let us point out that the Λ-FD is defined as

$$ {}_a^{\Lambda}D_x^{\gamma} f(x) = \frac{d\, {}_aI_x^{1-\gamma} f(x)}{d\, {}_aI_x^{1-\gamma} x} = \frac{dF(X)}{dX} \tag{13}$$

It will be clarified in the application, how from the initial space (x, f(x)) the fractional Λ-space (X, F(X)) is defined.

Further the pull back of the results in the initial space will also be indicated.

Geometry In The Λ-Fractional Space

Just to understand what happens in the Λ-fractional space, the geometry of the surface,

z=x2y2,0<x<1,0<y<1 (14)

will be discussed.

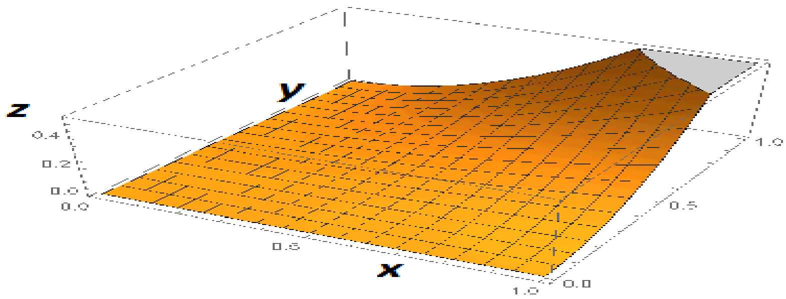

Figure1. *The surface z*

The fractional Λ-space (X,Y,Z) is defined by,

$$X = \frac{x^{2-\gamma}}{\Gamma(3-\gamma)} \tag{15}$$

$$Y = \frac{y^{2-\gamma}}{\Gamma(3-\gamma)} \tag{16}$$

$$Z = {}_bI_y^{1-\gamma}\, {}_\alpha I_x^{1-\gamma} z(x,y) = \frac{1}{(\Gamma(1-\gamma))^2} \int_b^y \left(\int_\alpha^x \frac{z(s,t)}{(x-s)^\gamma} ds\right) \frac{dt}{(y-t)^\gamma} \tag{17}$$

With a=b=0 , Eq.(17) yields,

$$Z = \left(-\frac{2\left(X\Gamma[3-\gamma]\right)^{\frac{1}{2-\gamma}}\right)^{3-\gamma}}{\Gamma[4-\gamma]} * Y\right) \tag{18}$$

For γ=0.6, the surface Z in the Λ- fractional space is defined by

Z=0.947X1.714Y1.714 (19)

and it is shown in Fig. 2.

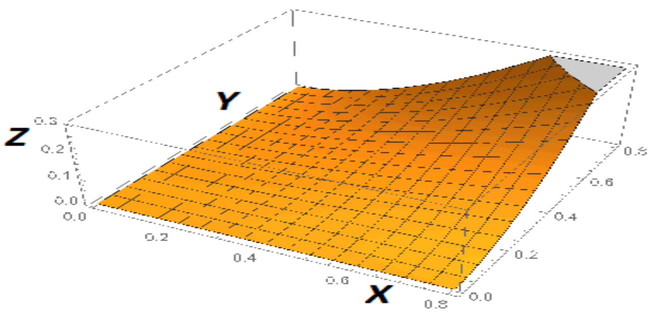

Figure2. *The surface Z in the Λ- fractional space*

Further, the tangent space of the surface with γ=0.6, at the point X=Y=0.6 is defined by,

$$Z = (0.947X^{1.714}Y^{1.714})_{(X=Y=0.6)} + \frac{dZ\,(X=Y=0.6)}{dX}(X-0.6) + \frac{dZ\,(X=Y=0.6)}{dY}(Y-0.6) \qquad (20)$$

and finally the equation of the tangent space in the Λ-fractional space,

$$Z=0.164+0.469(X-0.6)+0.469(Y-0.6) \qquad (21)$$

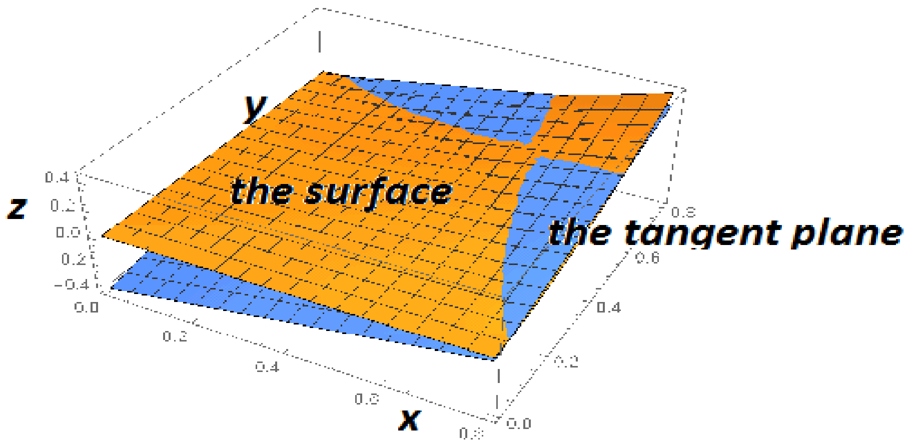

Figure3. *The surface with the tangent space in the Λ-fractional space*

The corresponding surface in the initial space to the tangent plane in the Λ-fractional space is defined by,

$$z= x^2y^2{}_{(x=y=0.81)} + \left({}^{RL}_{0}D^{1-\gamma}_{y=0.81}\ {}^{RL}_{0}D^{1-\gamma}_{x=0.81}\left(\frac{dZ}{dX}\right)\right)(X(x)-0.6)) + \left({}^{RL}_{0}D^{1-\gamma}_{y=0.81}\ {}^{RL}_{0}D^{1-\gamma}_{x=0.81}\left(\frac{dZ}{dY}\right)\right)(Y(y)-0.6)) \qquad (22)$$

The surface defined by Eq.(22) is shown in Fig. 4

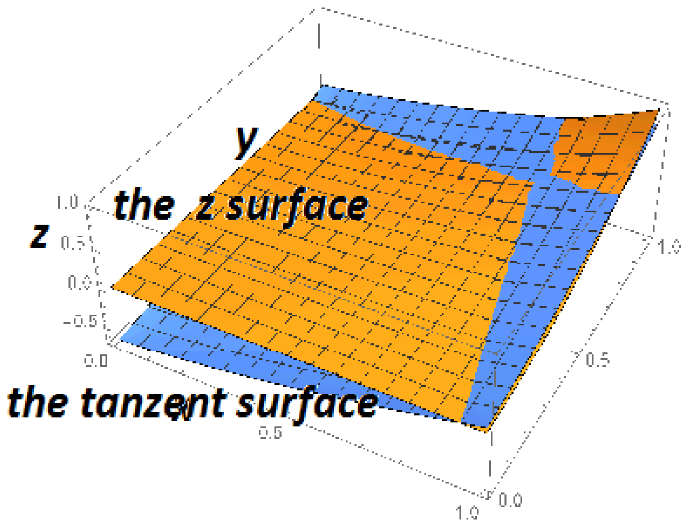

Figure4. *The surface with its tangent surface at the point (x=y=0.8106) at the initial space*

It seems that the initial surface and the tangent surface corresponding to the tangent space at the Λ-space have almost common tangent plane in the initial space.

The Fractional Field Theorems

The conventional field theorems are expressed by:

Green's Theorem

Let Qx(x,y), Qy(x,y),be smooth real functions in a domain Ω, with its boundary a smooth closed curve $\partial\Omega$. Then,

$$\int_{\partial\Omega}(Q_x dx + Q_y dy) = \iint_{\Omega} dxdy\left(\frac{dQ_x}{dy} - \frac{dQ_y}{dx}\right) . \qquad (23)$$

Corollary

When Qx(x,y), Qy(x,y), are derived by a potential function Φ(x,y) with $Q_x = \frac{d\Phi}{dx}, Q_y = \frac{d\Phi}{dy}$, the RHS of Eq (23) becomes zero.

That means that the curvilinear integral along a closed smooth boundary is zero.

Stoke's Theorem

For a smooth vector field F defined on a simple surface Ω with the boundary ∂Ω, Stoke's theorem is expressed by,

$$\int_{\partial\Omega} (\mathrm{F}, \mathrm{dL}) = \iint_{\Omega} (\nabla \times \mathrm{F}, \mathrm{dS}) \qquad (24)$$

where, (·,·) denotes the scalar product.

The Gauss' (Divergence) Theorem

For a space region Ω with smooth surface boundary ∂Ω, the volume integral of a the divergence of a vector field F over Ω is equal to the surface integral of F over the boundary ∂Ω:

$$\int_{\partial\Omega} (\mathrm{F}, \mathrm{dS}) = \iint_{\Omega} \nabla \cdot \mathrm{F}\, \mathrm{d}\Omega \qquad (25)$$

Although the field theorems are valid in the fractional Λ-space, they are not necessarily valid in the initial space. Nevertheless the results from the Λ- space may be pulled back to the initial space. The application that follows indicates the procedure.

Application (Green's Theorem)

Let us consider a rectangle with lx=1, ly=0.5. That rectangle is configurated in the fractional Λ-space with, Eq.(15,16):

$$L_x = \frac{{l_x}^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} \quad , \quad L_y = \frac{{l_y}^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} \qquad (26)$$

For γ=0.6 , Lx=0.8050, Ly=0.3050 . Further considering in the potential function,

Φ(X,Y)=X3Y2 (27)

The

$$Q_x = \frac{\partial\Phi}{\partial x} = 3X^2Y^2 \quad , \quad Q_y = \frac{\partial\Phi}{\partial y} = 2X^3Y \qquad (28)$$

According to the corollary of the Green's theorem, the curvilinear integral of Eq.(23) is zero in the Λ-space. Considering Eqs.(15,16)

$$Q_x = \frac{3\, x^{4-2\gamma} y^{4-2\gamma}}{\left((2-3\gamma+\gamma^2)\Gamma(1-\gamma)\right)^4} \quad , \quad Q_y = \frac{2\, x^{6-3\gamma} y^{2-\gamma}}{\left((2-3\gamma+\gamma^2)\Gamma(1-\gamma)\right)^4} \qquad (29)$$

Further, the corresponding functions in the initial space (x,y,z) are:

$$q_x = {}^{RL}_{\ 0}D_y^{1-\gamma}\left({}^{C}_{0}D_x^{1-\gamma}(Q_x(x,y)\right), \qquad qy = {}^{RL}_{\ 0}D_y^{1-\gamma}\left({}^{C}_{0}D_x^{1-\gamma}(Q_y(x,y)\right) \qquad (30)$$

where, Riemann-Liouville fractional derivative is given by Eq.(3). Considering γ=0.6 and performing the algebra with the help of Mathematica computerized algebra pack [20], we get

qx=3.124 x2.4y2.4 , qy=1.906x3.8y (31)

Although qx, qy may be derived by a conventional potential function, the fractional curvilinear fractional integral along the boundary of the rectangle is different from zero. In fact computing the fractional curvilinear integral with γ=0.6 along the boundary of the rectangle with 0<x<1 and 0<y<0.5 we get,

$$I_c^{\gamma} = 0.14564 \neq 0 \qquad (32).$$

Therefore the fractional Green's formula is not valid in the initial space. However, it is valid in the fractional Λ- space, where the fractional analysis and the fractional geometry follow the conventional rules.

Fractional Multiple Integrals And Calculus Of Variations

Since in the fractional Λ-space everything behaves conventionally, the variations of multiple integrals follow the well known common procedure, Weinstock [21]. Hence for a double integral in the fractional Λ-space,

$$I = \iint_{\Omega} L(X, Y, W, W_X, W_Y)\, dXdY \qquad (33)$$

yields the extremizing function,

$$\frac{\partial L}{\partial W}-\frac{\partial}{\partial X}\left(\frac{\partial L}{\partial W_X}\right)-\frac{\partial}{\partial Y}\left(\frac{\partial L}{\partial W_Y}\right)=0 \qquad (34)$$

along with the condition,

$$\frac{\partial L}{\partial W_X}\frac{dY}{dX}-\frac{\partial L}{\partial W_Y}=0 \qquad (35)$$

On the boundary C.

Application

The vibrating string .Following Weinstock [21] the equation of motion for a string is defined by Hamilton's principle extremizing the

$$I=\int_{t_1}^{t_2}(\widetilde{T}-V)dt=\frac{1}{2}\int_{t_1}^{t_2}\int_0^L\left(\sigma\frac{\partial^2 y}{\partial t^2}-\right.$$
$\tau\partial 2y\partial x2$ dxdt (36)

where σ(x) the density per unit length and τ is the tension along the string. Then the equation of motion of the string is defined by, see Weinstock [21],

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2} \qquad (37)$$

where $a2=\frac{\tau}{\sigma(x)}$.

For the present case a string, infinitely long has one end at x=0. The string initially rests on the x –axis. The end x=0 is subjected to transverse displacement that in the Λ-space is given by, A0sin(ωT). Find the displacement of any point on the string at any time.

Solution: If Y(X,T) is the transverse displacement of the string, then the boundary-value problem is defined by,

$$\frac{\partial^2 Y}{\partial T^2}=a^2\frac{\partial^2 Y}{\partial X^2} \qquad (38)$$

with the boundary and initial conditions

Y(X,0)=0,YT(X,0)=0,Y(0,T)=A0sint(ωT), (39)

with Y(X,T) bounded and a constant. For having a constant Σ(X) corresponding to σ(x) should be constant. Indeed, if

$$\sigma(x)=\frac{\delta}{{}_0I_x^{1-\gamma}1}=\delta\,\Gamma(2-\gamma)x^{1-\gamma} \qquad (40)$$

It is evident that the zero initial conditions in the Λ-space correspond to zero ones in the initial space (x, y, t). However the boundary condition in the Λ- space corresponds to the boundary condition in the initial space, defined by:

$$y(0,t)={}_0^C D_t^{1-\gamma}\left(A_0\sin\left(\omega\frac{t^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)}\right)\right)$$
(41)

Where according to Eq.(3), the Caputo derivative is equal to:

$${}_0^C D_t^{1-\gamma}f(t)=\frac{1}{\Gamma(\gamma)}\int_0^t\frac{f(s)-f(0)}{(t-s)^{1-\gamma}}dt \qquad (42)$$

For A0=ω=1 and γ=0.6, the boundary condition y(0,t) , Eq.(41), has been computed, in the initial space, with the help of the Mathematica pack and is shown in Fig. 5

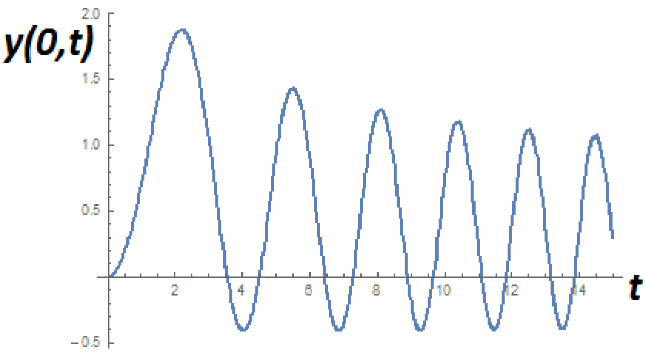

Figure5. *The initial condition in the initial space*

The problem has been solved in the fractional Λ-space using Laplace's transformation and the solution may be found in [22] p.224. The solution is defined through Laplace's transformation as:

Y(X,T)= A0sin(ωT-X/a) for T>X/a (43)
= 0 for T<X/a

Let us remind that , Eq.(9),

$$X = \frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} = 0.805\, x1.4\,,$$

$$T = \frac{t^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} = 0.805t1.4 \qquad (44)$$

Hence,

Y(x,t)=A0sin(0.805(ωt1.4-x1.4/a)) (45)

According to the theory, the displacement of the string in the initial space is defined by,

$$y(x,t) = {}_0^C D_t^{1-\gamma}\, {}_0^C D_x^{1-\gamma} Y(x,t) \qquad (46)$$

Furthermore, the displacement y(x,t) has been computed trough the Mathematica computerized pack and is shown in Fig.6 with A0=ω=a =1 and γ=0.6.

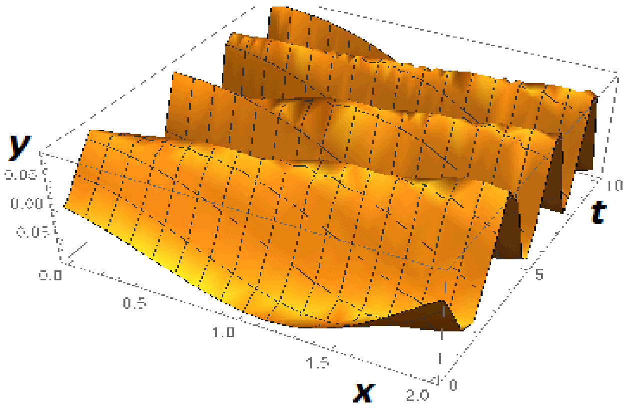

Figure6. *The displacement y(x,t) of the string*

Conclusion

The fractional field theorems have been discussed in the context of the fractional Λ-derivative and the corresponding fractional Λ-space. Since the fractional Λ- derivative behaves in the conventional way in the Λ- space, the analysis concerning the geometry and the field theorems is developed in the Λ-space. The results are transferred back to the initial space through the Caputo derivative. Furthermore, variational methods may be applied in the Λ-fractional space in the conventional way and then the results may be pulled back to the initial space.

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Citation: *K.A.Lazopoulos, " On Λ-Fractional Field Theorems", International Journal of Research Studies in Science, Engineering and Technology, vol. 8, no. 1, pp. 30-37, 2021, DOI : https://doi.org/ 10.22259/2349-476X.0801004*