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On the Convergence of Derivative of Hermite Interpolating Polynomial

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Abstract: *In this paper, we consider the zeros of* $(1\text{-}x^2)P_n^{(\alpha,\beta)}(x)$, *where* $P_n^{(\alpha,\beta)}(x)$ *stands for Jacobi polynomial, which are vertically projected onto the unit circle. Here, we are interested to establish the convergence theorem for the derivative of the Hermite interpolatory polynomial on the above said nodes.*

Mathematics subject classification: 41A05, 30E10.

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1. Introduction

In a paper, J.Szabad os [8] treated the convergence problems of the derivatives for arbitrary projection operators on C [-1,+1]. In another paper, K.Balázs and T.Kilgore [3] considered the approximation of derivatives by interpolation. Also, G. Mastroianni [5] investigated the uniform convergence of derivatives of Lagrange interpolation at the union of the zeros of Jacobi polynomials and some additional points. Also, G. Mastroianni and P. Nevai [6] established mean convergence of derivatives of Lagarange interpolation. In a paper, K. Balázs [2] proved the simultaneous convergence of the derivatives of Lagarange interpolating polynomials by giving an estimate for $\left|f^{(i)}(x) - L_n^{(i)}(f,x)\right|$ $(i = 0,1,2, \dots \dots \dots)$by the aid of the Lebesgue constant of Lagrange interpolation. The above paper has motivated us to consider convergence of the derivative of the Hermite interpolation on the unit circle.

In this paper, we consider the derivative of the Hermite- interpolation on the vertically projected zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial. In section 2, we give some preliminaries and in section 3, we describe the problem. In section 4, we give the explicit formulae of the interpolatory polynomials. In sections 5 and 6, estimation and convergence of interpolatory polynomials are considered respectively.

2. Preliminaries

In this section some well-known results are given, which we shall use.

The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is

$$(2.1)\qquad (1-x^2)P_n^{(\alpha,\beta)''}(x)+[\beta-\alpha-(\alpha+\beta+2)x]P_n^{(\alpha,\beta)'}(x)+n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0$$

$$(2.2)\qquad W(z)=\prod_{k=1}^{2n}(z-z_k)=K_nP_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^n$$

$$(2.3)\qquad R(z)=(z^2-1)W(z)$$

We shall require the fundamental polynomials of Lagrange interpolation based on the nodes as zeros of R(z) and W(z) are given by:

$$(2.4)\qquad L_k(z)=\frac{R(z)}{R'(z_k)(z-z_k)},k=0(1)2n+1$$

We will also use the following results

(2.5) $$(-1)^n W'(z_{n+k}) = W'(z_k) = -\frac{1}{2} K_n P_n^{(\alpha,\beta)'}(x_k)(1-z_k^2) z_k^{n-2}, k = 1(1)n$$

(2.6) $$(-1)^{n-1} W''(z_{n+k}) = W''(z_k) = -\frac{1}{2} K_n P_n^{(\alpha,\beta)'}(x_k) \begin{bmatrix} -\begin{pmatrix} 2(\beta-\alpha) z_k \\ -(\alpha+\beta+2)(1+z_k^2) \end{pmatrix} \\ + 2n(1-z_k^2) - 2 \end{bmatrix} z_k^{n-3}, k = 1(1)n$$

(2.7) $$R'(z_k) = (z_k^2 - 1) W'(z_k)$$

(2.8) $$R''(z_k) = W'(z_k)\left[4z_k^2 + 2(\beta-\alpha) z_k - (\alpha+\beta+2)(1+z_k^2) - 2n(1-z_k^2) + 2\right] z_k^{-1}$$

We will also use the following well known inequalities (see [9])

(2.9) $$(1-x^2)^{\frac{1}{2}} P_n^{(\alpha,\beta)}(x) = o(n^{\alpha-1}) \text{ for } \alpha > 0, x \in [-1,1]$$

(2.10) $$(1-x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}$$

(2.11) $$\left|P_n^{(\alpha,\beta)'}(x_k)\right| \sim k^{-\alpha-\frac{3}{2}} n^{\alpha+2}$$

(2.12) $$\left|P_n^{(\alpha,\beta)}(x)\right| = o(n^{\alpha}), \alpha > 0$$

(2.13) $$\left|P_n^{(\alpha,\beta)'}(x)\right| = o(n^{\alpha+2})$$

3. THE PROBLEM

Let $Z_n = \{z_k : k = 0(1)2n+1\}$ satisfying:

$$Z_n = \{z_0 = 1, z_{2n+1} = -1,$$
$$z_k = \cos\theta_k + i\sin\theta_k, z_{n+k} = -z_k, k = 1(1)n\}$$

be the vertical projections on the unit circle of the zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$, where $P_n^{(\alpha,\beta)}(x)$ stands for the Jacobi polynomial having the zeros $x_k = \cos\theta_k$, k=1 (1)n such that $1 > x_1 > \cdots \ldots \ldots \ldots .. > x_n > -1$ .In this paper, we shall use the polynomials $H_n(z)$ of lowest possible degree satisfying the conditions

(3.1) $$\begin{cases} H_n(z_k) = \alpha_k, & k = 0(1)2n+1 \\ H'_n(z_k) = \beta_k, & k = 0(1)2n+1 \end{cases}$$

where α_k and β_k are arbitrary given complex numbers.

$H_n(z)$ is known as Hermite interpolation on unit circle [1]. In this paper, we seek to determine the convergence of the derivative of $H_n(z)$ on the above said nodes.

4. EXPLICIT REPRESENTATION OF INTERPOLATORY POLYNOMIALS

We shall write $H_n(z)$ satisfying (3.1) as:

(4.1) $$H_n(z) = \sum_{k=0}^{2n+1} \alpha_k A_k(z) + \sum_{k=0}^{2n+1} \beta_k B_k(z)$$

Where $A_k(z)$ and $B_k(z)$ are fundamental polynomials of the first and second kind respectively, each of degree at most 4n+3 satisfying the conditions:

For j, k = 0(1)2n+1

$$(4.2)\qquad \begin{cases} A_k(z_j) = \delta_{jk} \\ A'_k(z_j) = 0 \end{cases}$$

$$(4.3)\qquad \begin{cases} B_k(z_j) = 0 \\ B'_k(z_j) = \delta_{jk} \end{cases}$$

THEOREM 2 [1]: For k=0(1)2n+1, we have,

$$(4.4)\qquad B_k(z) = \frac{\boldsymbol{R(z)L_k(z)}}{\boldsymbol{R'(z_k)}}$$

THEOREM 3 [1]: For k=0(1)2n+1, we have,

$$(4.5)\qquad A_k(z) = L_k^2(z) - 2L'_k(z_k)B_k(z)$$

5. Estimation of Fundamental Polynomials

LEMMA 1 [1]: Let $L_k(z)$ be given by (2.4). Then

$$(5.1)\qquad \max_{|z|=1} \sum_{k=0}^{2n+1} |L_k(z)| \le \frac{c}{k^{-\alpha+\frac{3}{2}}}$$

where c is a constant independent of n and z.

LEMMA 2: Let $L_k(z)$ be given by (2.4). Then

$$(5.2)\qquad \max_{|z|=1} \sum_{k=0}^{2n} |L'_k(z)| \le \frac{cn}{k^{-\alpha+\frac{1}{2}}}$$

Where c is a constant independent of n and z.

PROOF: Also,

$$(5.3)\qquad \begin{cases} z_k = x_k + iy_k \\ |z^2-1| = 2\sqrt{1-x^2} \\ |z_k^2 - 1| = 2\sqrt{1-x_k^2} \\ |z - z_k| = \sqrt{2}\sqrt{1 - xx_k - \sqrt{1-x^2}\sqrt{1-x_k^2}} \end{cases}$$

Differentiating (2.4), we get

$$(5.4)\qquad \sum_{k=0}^{2n} |L'_k(z)| = \sum_{k=0}^{2n} \left| \frac{R'(z)(z-z_k) - R(z)}{R'(z_k)(z-z_k)^2} \right|$$

Using (2.2), (2.3), (2.5), (2.7) and (5.3) in (5.2) and further using (2.9)-(2.13), we get required equation (5.2).

LEMMA 3: Let $B'_k(z)$ be obtained by differentiating (4.4). Then

$$(5.5)\qquad \sum_{k=0}^{2n} |B'_k(z)| < cn\log n \quad , \quad \alpha \le -\frac{1}{2}$$

where ,c is a constant independent of n and z.

PROOF: Differentiating (4.4), we get

$$\sum_{k=0}^{2n} |B'_k(z)| = \sum_{k=0}^{2n} \left| \frac{R'(z)\, L_k(z) + R(z)\, L'_k(z)}{R'(z_k)} \right|$$

Using (2.2), (2.3), (2.5) and (2.7) in (5.6), we get

$$\sum_{k=0}^{2n}|B_k'(z)|$$
$$\leq \sum_{k=0}^{2n} \frac{\left| 2z\,K_n P_n^{(\alpha,\beta)}(x) z^n + \sqrt{1-x^2}\begin{bmatrix}\sqrt{1-x^2}\,\frac{1}{2}K_n P_n^{(\alpha,\beta)'}(x_k)+ \\ n\,K_n P_n^{(\alpha,\beta)}(x) z^{n-1}\end{bmatrix}\right| |L_k(z)| + \left|K_n P_n^{(\alpha,\beta)}(x) z^n \sqrt{1-x^2}\right| |L_k'(z)|}{\left|(1-x_k^2)\,P_n^{(\alpha,\beta)'}(x_k)\right|}$$

Further, using (2.9), (2.10), (2.11), (2.12), (2.13), Lemma 1 and Lemma 2, we get (5.5).

LEMMA 4: Let $A_k'(z)$ be obtained by differentiating (4.5). Then

(5.6) $\sum_{k=0}^{2n}|A_k'(z)| < cn^2 \log n \quad , \ \alpha \leq -\frac{1}{2}$

where c is a constant independent of n and z.

PROOF: Differentiating (4.5), we get

$$\sum_{k=0}^{2n}|A_k'(z)| \leq \sum_{k=0}^{2n}|2L_k(z)L_k'(z)| + \sum_{k=0}^{2n}\left|\frac{R''(z_k)}{R'(z_k)}\right| |B_k'(z)|$$

Using Lemma 1, Lemma 2 and Lemma 3, we get (5.6)

6. CONVERGENCE

Let *f(z)* be analytic for $|z|<1$ and continuous for $|z|\leq 1$ and $\omega_2(f,\delta)$ be the modulus of continuity of $f(e^{i\theta})$.

THEOREM 4: Let *f (z)* be continuous in $|z|\leq 1$ and analytic in $|z|<1$. Let the arbitrary numbers β_k 's be such that:

(6.1) $|\beta_k| = o\left(n\omega_2(f,n^{-1})\right)$

Then $\{H_n\}$ be defined by:

(6.2) $H'_n(z) = \sum_{k=0}^{2n} f(z_k)\,A'_k(z) + \sum_{k=0}^{2n} \beta_k\,B'_k(z)$

satisfies the relation:

(6.3) $\left|H_n'(z) - f'(z)\right| = o\left(n^2\omega_2(f,n^{-1})\log n\right) \ , \ \alpha \leq -\frac{1}{2},$

where $\omega_2(f,n^{-1})$ is the modulus of continuity of *f(z).*

REMARK 1: Let *f(z)* be continuous in $|z| \leq 1$ and $f' \epsilon\, Lip\ \alpha$, α>1, then the sequence $\{H_n'\}$ converges uniformly to *f'(z)* in $|z| \leq 1$, follows from (6 .3) provided

$$\omega_2(f,n^{-1}) = o(n^{-1-\alpha})$$

To prove theorem 4, we shall need the following:

REMARK2: Let *f (z)* be continuous in $|z|\leq 1$ and analytic in $|z|<1$ .Then there exists a polynomial $F_n(z)$ of degree 4n+1 satisfying Jackson's inequality

(6.4) $\left|F_n'(z) - f'(z)\right| \leq c\omega_2(f,n^{-1}) \quad , \quad z = e^{i\theta}(0 \leq \theta < 2\pi)$

And also an inequality due to O.Kiš [4]

$$(6.5) \qquad \left|\mathrm{F}_{\mathrm{n}}^{(\mathrm{m})}(z)\right| \le cn^{m}\omega_2\left(f, n^{-1}\right) \quad , \quad \text{for } \mathrm{m} \in \mathrm{I}^{+}$$

PROOF: Since $H_n'(z)$ be given by (6.2) is a uniquely determined polynomial of degree $\le 4n+1$, the polynomial $F_n'(z)$ satisfying (6.4) and (6.5) can be expressed as :

$$F_n'(z) = \sum_{k=0}^{2n} F_n(z_k) A_k'(z) + \sum_{k=0}^{2n} F_n'(z_k) B_k'(z)$$

Using $z = e^{i\theta}\left(0 < \theta \le 2\pi\right)$, (5.5), (5.6), (6.1),(6.4),(6.5) and Lemma 3 and Lemma 4, we get (6.3).

Hence, the theorem follows.

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