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Contributions of Euler, Gauss and Riemann to the Study of Primes Numbers

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Abstract: *Numbers theory is an important branch of mathematics; having produced abundant literature since the nineteenth century, the majority of which being related to the study of prime numbers. In the history of science two individuals of note, namely Euler and Gauss, made significant contributions to the explanation of these numbers and these contributions lead up to a watershed marking the before and after of the analytic mathematical development on the subject known as Riemann Hypothesis (RH), which dates back to 1859 and has yet to be sufficiently proven. We used MATLAB and MATHEMATICA to reproduce certain graphs that help us come to a better understanding. To show pattern irregularities the arbitrary behavior of these numbers is plotted. Next we start with the product of Euler and the zeta function; following that, we explain the prime number theorem of Gauss and the logarithmic function; and finally, we present Riemann Hypothesis (RH), the zeros of its function, and the critical line. To focus on these three individuals of note presents some clarity around this problem, which is an important part of on-going research regarding subjects such as quantum chaos, cryptography, and computer.*

Keywords: *Critical line, Mobius function, fundamental principle of arithmetic, Gamma function*

1. Introduction

Over the past decade, books and articles regarding the subject have been published. Of special interest is the book of Marcus du Sautoy [1], a book vital for those interested in the historical development of the subject in that it stresses anecdotal facts and provides a complete analysis due to its coverage of recent investigations to solve the Riemann hypothesis using quantum physics. This underscores that currently there is a convergence of numbers theory, physics, and chaos theory. Another great teaching resources is that of Germán Sierra [2], a professor who provides a very convincing explanation of the relationship between physics and prime numbers. Essential too is the book by Enrique Gracian [3] with more mathematical content and noted for the high quality of its teaching methods that help students grasp the necessary concepts to understand Riemann hypothesis.

For this research we consulted important literature. First, we studied an article from Felipe Zaldivar [4] that includes detailed mathematical developments as well as an appendix with Riemann's original work. José Manuel Sánchez Muñoz [5] wrote an article that covers the historical development of prime numbers including mathematical analysis, even after RH is established. In fact, it describes the contributions of Ramanujan, Hardy and Littlewood who came later than Riemann and who are known as the triplet of Trinity College, because they revived England from a very prolonged mathematical lethargy [1].This triplet have great contributions that require focused analysis.

About Euler literature there are many such books that are highly recommended [6] and [7] with good mathematical content. The literature about Gauss is almost nonexistent, however there are available [8] and [9], this latter text is not specific about Gauss, but contains lots of good information from the entire chronographic history of mathematics.Also about the life and work of Riemann we have [10] with an accurate narrative of the contribution of this revolutionary man of the exact sciences.Like all works are scattered, we think that there are new ways to study the number theory based on this essential trick, in the scientific development of mankind.

Our work searches to focus on the achievements of Euler and Gauss in order to establish a new method of reaching the hypothesis of Riemann using programs such as Matlab and Mathematica. We generated the theorem of prime numbers, the integral logarithmic function, the graph of the complex solution of Riemann hypothesis and the mount of Riemann for an arbitrary size, in order to solidify the link that prime numbers have with Riemann Hypothesis. For last, an analysis of the approach employed was made in an effort to clarify its validity as a legitimate method to study the subject..

2. Nature and Randomness of Prime Numbers

Prime numbers are known as the atoms of arithmetic, because they can construct all integers that are not prime. It is well known as a complicated area of study given its lack of order and pattern, thus representing a major mathematical challenge. One of the most celebrated topics of mathematicians is Riemann Hypothesis on the distribution of primes, where their chaos is changed into an ordered system. To demonstrate Riemann hypothesis implies the completion of the theorems that depend on its accuracy, that is, which can transform the conjecture into a theorem. It also confirms many other theorems that depend on the accuracy of RH. Figure 1 shows the capricious and arbitrary form in which the first one hundred prime numbers are distributed.

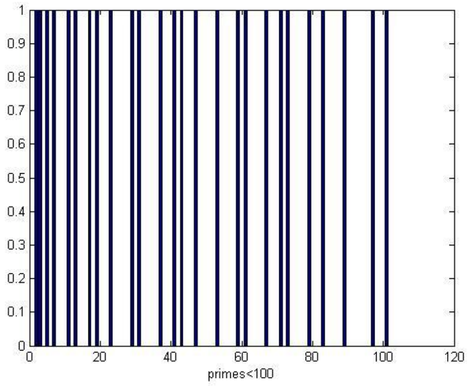

Figure 1. *Lack of pattern or model*

3. Riemann Hypothesis Background

The real forgers of the legend inherited by Riemann were directly Dirichlet and Gauss, who were his teachers at Göttingen, but at that time during the nineteenth century its background stemmed from two iconic mathematicians, Euclides and Euler, and not be overlooked were others such as Fermat, Marsenne, Goldbach, and other historical giants who courageously contributed to the understanding of prime numbers. Euclid's groundbreaking theorem asserts that "all natural numbers can be decomposed as products of prime factors", which is also part of those considered infinite, and it is worth mentioning that as they grow, they become fewer making its estimation more difficult.

Many are the true scientific feats of Leonardo Euler (1707-1783) and prime numbers did not escape from his field of study; he defined a function based on the harmonic series.

$$\zeta(x) = \frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + ... + \frac{1}{n^x} = \sum_{n=1}^{\infty} \frac{1}{n^x} \tag{1}$$

Euler found that this was related to the number π, the squaring of the circle, for example

$$\zeta(2) = \frac{\pi}{6}, \quad \zeta(4) = \frac{\pi}{90} \tag{2}$$

Euler established a relationship between this function and prime numbers. It has been said that with this discovery the analytic study of prime numbers began.

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} = \prod_p \frac{1}{1 - p^{-x}} \tag{3}$$

Where p is prime number, Euler wrote in 1751 "there are some mysteries which the human mind will never penetrate, to persuade us to see the tables of primes" [1].Later he discovered the equation (3) to advance the understanding of the matter, however almost one hundred years have topass another until another mind such Riemann could interpret the fact;nevertheless, it was Gauss who encouraged this last oneto continue another perspective.

Prime number theorem is attributed to Friedrich Gauss (1777-1855), the prince of mathematics, who discovered that the randomness of prime numbers follows a definite order. He found that there is an average law and defined a function such that $\pi(x)$ is the amount of prime numbers smaller than x, consequently

$$\pi(10) = 4; \quad \pi(15) = 6$$

The Table 1 shows the first values of the function.

Table 1. *First function values.*

x	10	100	1000	10,000	100,000
$\pi(x)$	4	25	168	1,229	9592
$\pi(x)/x$	0.400	0.250	0.168	0.1229	0.0952

In the third row of the table, it can be seen how this decreases as they get closer to larger numbers; Gauss found a relationship with natural logarithm

$$\frac{\pi(x)}{x} = \frac{1}{\ln x} \tag{4}$$

Independently, Legendre and Gauss established that

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1 \tag{5}$$

Figures 2 and 3 show the theorem of prime numbers for different orders of magnitude

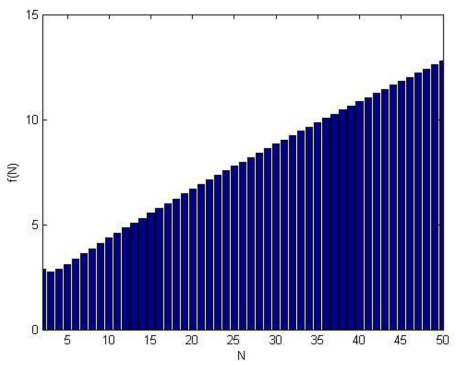

Figure 2.*Theorem or prime numbers, N<100*

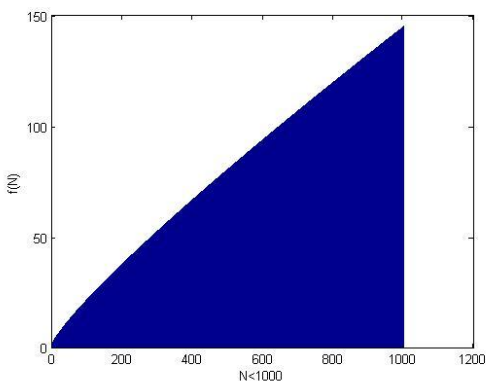

Figure 3.*Theorem or prime numbers, N<1000*

The advantage was to have this development stated in another way, that is by creating the function $Li\ \ N$, which is more precise

$$\frac{1}{\log(2)}+\frac{1}{\log(3)}+\frac{1}{\log(4)}+\cdots\frac{1}{\log N} \tag{6}$$

Such that

$$\sum_{2\le n\le x}\frac{1}{\log n}=\int_2^x\frac{dt}{\log t} \tag{7}$$

$$\pi\ \ x\ \ \approx Li=\int_2^x\frac{dt}{\log t} \tag{8}$$

Figure 4 shows the integral logarithmic function

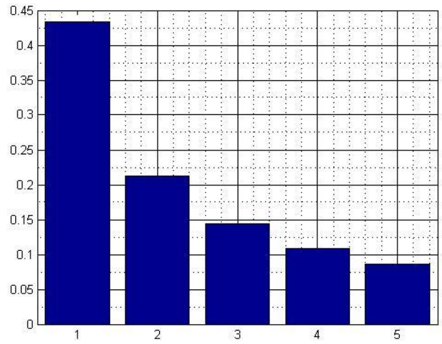

Figure 4.*Integral logarithmic function.*

In generalGauss published little, nevertheless, his mathematical papers in various fields are of such profundity, that they have influenced the progress of the subject to the present day [9]. The*disquisitiones arithmeticae*published in 1801 represent the beginning of the study of the theory of numbers.

4. Riemann Hypothesis

Bernhard Riemann (1826-1866) studied Gauss' conjecture and proposed to make a zeta function for complex values, which became Riemann's zeta function

$$\zeta\ \ s\ \ =\sum_{n=1}^{\infty}\frac{1}{n^s}\ ,\quad s=a+bi\ \operatorname{Re}\ s>1 \tag{9}$$

Bernhard Riemann was Gauss' student. He asked him to develop the theorem of prime numbers. In 1859 he wrote a publication about the topic. In this article, the famous Riemann Hypothesis is included. During the following 30 years after his death, little progress was made regarding the problems of the distribution of primes. It was in 1901 when Helge von Koche found that

$$\pi\ \ x\ \ =Li\ \ x\ \ +O\ \ \sqrt{x}\ln x \tag{10}$$

Where O is the o of Landau. Other famous mathematicians related to this are J. Hadamard and C. J. de la Vallée-Poussin, who individually demonstrated in 1896, the primary formula to distribute these arbitrary numbers which is known as prime number theorem; conjectured a century before by Gauss and Legendre. These authors did something very significant since they found relationships between the zeta function and the logarithmic function proposed by Gauss in the following way

$$\pi(x) = Li(x) + O\left(\frac{x}{\log(x)} e^{-a\sqrt{\log(x)}}\right) \tag{11}$$

RH has resulted in linking the Gauss Theorem in the following way

$$\left|\pi(x) - \int_0^x \frac{dt}{\ln(t)}\right| \leq C\sqrt{x}\ln(x) \tag{12}$$

Where $C = \frac{1}{8\pi}$ $\quad x \geq 2657$

Prime numbers and zeros of the zeta function give rise to the same space, that is, there is a strong connection between them, an issue that was foreseen from the product of Euler. Likewise, Gauss had a function that tells us that there are N primes in an amount smaller than N. Riemann proposed $R(N)$ which estimates the primes according to the following table 2.

Table 2.

N	primes(N)	$R(N)$	$Li(N)$
10^2	25	26	30
10^3	168	168	178
10^4	1229	1227	1236
10^5	9592	9587	9630

Where we see a notorious improvement, such function is written as follows

$$R(N) = \sum_{r=1}^{\infty} \frac{\mu(r)}{r} Li\left(N^{1/r}\right) \tag{13}$$

Where $\mu(r)$ is the Mobius function such that

$$\mu(r) = \begin{cases} 0, & r \text{ has repeated factor} \\ 1, & r \text{ has an even number of primes factor} \\ -1, & r \text{ has an odd number of primes factor} \end{cases} \tag{14}$$

RH can be written in other ways, one is called the integral formula of the zeta function which is written as follows

$$\varsigma(s) = \frac{i}{2\pi}\Gamma(s-1)\int_C \frac{(-z)^{z-1}}{e^z - 1} dz \quad s \in C \tag{15}$$

A second expression, the one called functional equation, is as follows [11]

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \tag{16}$$

Where $\Gamma(1-s)$ is the known Gamma function. If we see the written function in this form we see that it has certain values called non-trivial zeros. For example $s = -2,\ s = -4, \ldots\ldots$. However, there are other complex values s between $0 < \text{Re}\ s < 1$ where the function is cancelled, which are called non-trivial zeros. Riemann's conjecture says: the real part of all non-trivial zeros of the zeta function is $1/2$. Riemann zeta function is deeply rooted with prime numbers and Helge von Koch demonstrated in 1901 that Riemann hypothesis is equivalent to the prime number theorem.

Riemann found a 3D image of the zeros of the zeta function where there is a critical stripe whose real values are located between 0 and 1 and that is related to prime numbers; using the program Mathematica we can obtain the behavior of the zeta function and its reciprocal. We plotted in figure 5

$$f\ x,y\ = |\zeta\ x+iy\ | \quad (17)$$

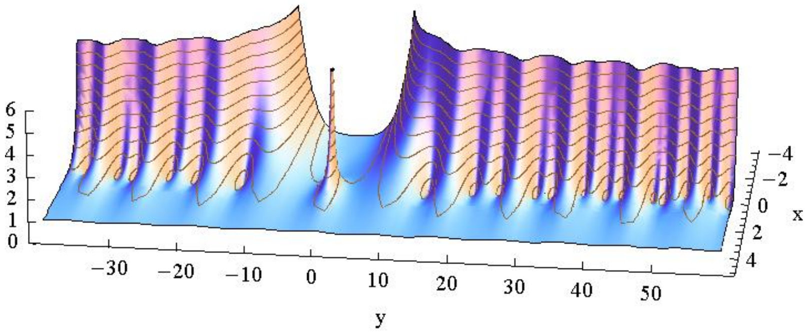

Figure 5. *Mount of Riemann*

The surface show that the graph with the singularity at 1 prominent. If we generate the reciprocal of $f\ x,y$ in figure 6, which brings out zeros more clearly.

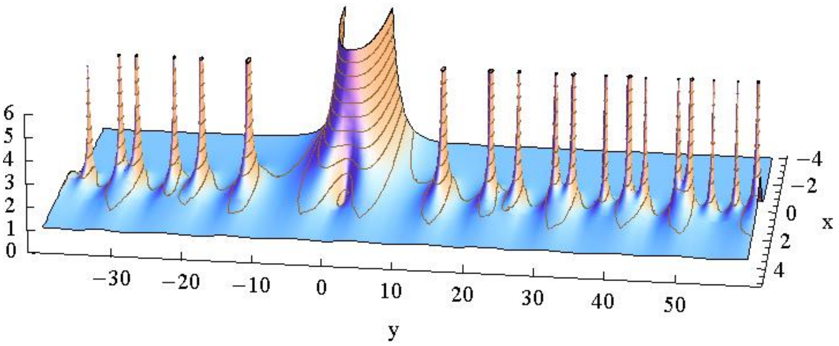

Figure 6. *Reciprocal Mount of Riemann*

The graphs shows thirteennontrivial zeros, all of which lie on the line is called critical line. The estimations of RH were congruent and agree with the result of Jacques Hadamard and Charles de la Vallée-Poussin, when they demonstrated the theorem of prime numbers expressed by Gauss. However, the main part of Riemann's work consists in saying that all non-trivial zeros of the zeta function can be located in the coordinate $\frac{1}{2}+iy$, which is equivalent to saying that the real part of all non-trivial zeros of the zeta function is $1/2$.

Next, we plot in Figure 5 the function $\zeta\ 0.5+ix$ where the first zeros of Riemann function are shown (14.13, 21.02, 25.01, 30.42, 32.93, 37.58)

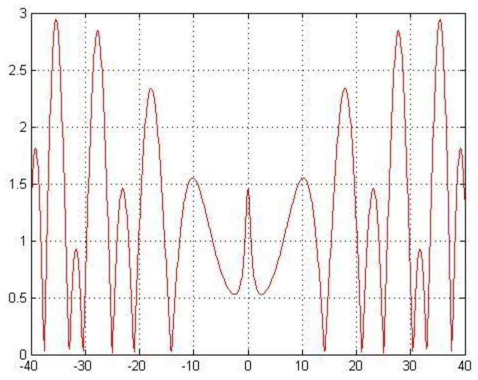

Figure 7.*Zeros of Riemann Zeta Function*

Additionally, with Mathematica it is possible to generate a polar graph and can help to explain critical line. We plot in figure 8 with polar coordinates

$$\zeta\left(\frac{1}{2}+it\right) \qquad 0<t<26 \tag{18}$$

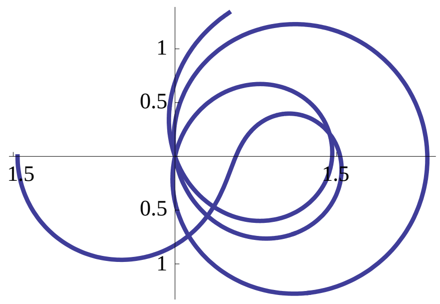

Figure 8.*Polar equation of the Riemann Zeta Function.*

The curve passes through the origin three times, corresponding to the first three zeros on the critical line, this graph has the called Gram Points, this are points on the critical line where the zeta function is real and non-zero.

In 1859 Riemann was admitted to the Academy of Sciences in Berlin, his presentation was to work on the distribution of prime numbers, the reference [4] presents a good Spanish translation of that famous work starts RH, his mathematical work was brillant and transcendent.

5. Discussion of Results

From the history of Euler, Gauss and Riemann much valuable teaching material can be generated and lessons drawn of analytical mathematical interest. For example, the product of Euler of equation (3) is easy yet at the same time important to carry out the demonstration and Matlab can assist with this. We must remember that Dirichlet and Ramanujan generalized this expression and, for the purpose of this paper, are equations that we are not presenting. In Gauss Prime Number Theory, equation (4), it is relatively simple to observe the relationship of primes with natural logarithm; in addition, it has great importance in Riemann hypothesis, as it is seen in equations (10) and (11). The logarithm function minimally improves the Gauss theorem and it is plotted in Figure (4). There are several forms to represent Riemann hypothesis. For example, equation (13) is the counter of prime numbers, while the functional representation of equation (16) is important to distinguish trivial zeros. It is crucial to make the mathematical developments necessary to obtain the integral expression and the functional equation, which can be confirmed in references [4] y [5]. The graph in 3D of Riemann zeta function in figures 5 and 6 has the purpose to check if the distribution of primes confirms the critical line that is established in this conjecture, while Figure 7 provides the roots that are associated to the pace of primes.

6. Conclusions

One of the virtues included in looking for the solution of RH is the possibility to converge areas that did not have contact before, such as the chaos theory, numbers theory, probability and quantum physics. Prime numbers are still an enigma and in the following years we expect important discoveries, mainly using quantum models to explain them. In this work we focused on showing the contributions of Euler and Gauss in order to put together Riemann hypothesis. We plotted in a graph the behavior of certain functions, where the integral logarithmic function and the theorem of prime numbers are emphasized. In addition, we highlighted the equations that indicate in summarized form the RH. However, we still have to include the mathematical developments that generate such equations and also the analysis of many functions involved such as the so-called arithmetic functions; for example, the Mobius function can undergo a numeric treatment. Additionally, although Riemann conjecture was numerically proved in the first ten trillions of non-trivial zeros, in mathematics this is not permitted as a strict demonstration. Reportedly, in numeric analyses, it is more interesting to find a counter example, that is, a zero value whose real part is not 1/2, which casts doubts on the genuineness of the conjecture. Our work shows a general outlook and describes the historical development. In addition, it shows how the program can be applied to a topic in number theory which today is studied based on the analyses of mathematical functions. Finally, it should be noted that Riemann conjecture is the cornerstone of and basis for further work, that is, it depends on its accuracy to complete many theorems. Therefore, the aforementioned gives us the opportunity to be able to remember key moments and here we want to locate our contribution through the compilation of the history that began with Euler, the most prolific mathematician in history and the first one to discover the relationship between the zeta function and primes. We include Gauss, another historically renowned mathematician, who based his theorem on the logarithm function as well as the fact that Riemann's revolution was passed on to future generations, the likes of which we would never unjustifiably compare with Dirichlet, Goldbach, and Fermat, who although having the academic merit, were never as influential as Euler, Gauss and Riemann in the mathematical tradition and their significance in the history of science; therefore we consider thathas value perform different math lessons. Our contribution was to study the numbers primes with graphical support and function to that unforgettable mathematical trick.. .

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